Online Appendix

"Core Deviation Minimizing Auctions"
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April 23, 2014

Proposition 3 *The core deviation minimization problem can be written as a standard calculus of variations problem of the form*

$$\min_r \int_0^2 A\left(r\left(v\right), r'\left(v\right), v\right) dv. \tag{1}$$

Proof of Proposition 3 Denote

$$\int_0^2 \left(\int_0^{\min\{v,1\}} \int_0^{\min\{v-v_1,1\}} (v_1+v_2-r\left(v_1+v_2\right))dF\left(v_2\right)dF\left(v_1\right) \right) dG\left(r\left(v\right)\right)$$

by O_1 and

$$\int_0^2 \left(\int_{\max\{v-1,0\}}^1 \int_{\max\{v-v_1,0\}}^1 \max\{r\left(v\right) - \max\{v-v_1,0\} - \max\{v-v_2,0\},0\}dF\left(v_2\right)dF\left(v_1\right) \right) dG\left(r\left(v\right)\right)$$

by O_2.
Denote the expectation of F by L,

$$L\left(v\right) = \int_0^v v' f\left(v'\right) dv'. \tag{2}$$

Moreover, denote v_1+v_2 by t and the distribution of t by H. That is

$$H\left(t\right) = \int_0^{\min\{t,1\}} \int_0^{\min\{t-v_1,1\}} dF\left(v_2\right)dF\left(v_1\right). \tag{3}$$

Also denote its density by h, $h\left(t\right) = H'\left(t\right)$, and expectation by K,

$$K\left(t\right) = \int_0^t t' h\left(t'\right) dt'. \tag{4}$$

Lastly, denote v_1+v_2 conditional on $v_1, v_2 < v$ by t_v and its distribution by H_v. That is,

$$H_v\left(t_v\right) = \int_0^{\min\{t_v,v\}} \int_0^{\min\{t-v_1,v\}} dF\left(v_2\right)dF\left(v_1\right). \tag{5}$$

Also denote its density by h_v, $h_v\left(t\right) = \frac{d}{dt}H_v\left(t\right)$, and expectation by K_v,

$$K_v\left(t\right) = \int_0^t t' h_v\left(t'\right) dt'. \tag{6}$$

Then, we can write

$$
\begin{aligned}
O_1 &= \int_0^2 \left(\int_0^v (t - r(t)) dH(t) \right) dG(r(v)) \\
&= \int_0^2 \left(K(v) - \int_0^v r(t)\, h(t)\, dt \right) dG(r(v)) \\
&= \int_0^2 K(v)\, dG(r(v)) - \int_0^2 \left(\int_0^v r(t)\, h(t)\, dt \right) dG(r(v)) \\
&= \int_0^2 K(v)\, dG(r(v)) - \int_0^2 \left(\int_t^2 dG(r(v)) \right) r(t)\, h(t)\, dt \\
&= \int_0^2 K(v)\, dG(r(v)) - \int_0^2 (1 - G(r(t)))\, r(t)\, h(t)\, dt \\
&= \int_0^2 \left(K(v)\, g(r(v))\, r'(v) - (1 - G(r(v)))\, r(v)\, h(v) \right) dv
\end{aligned}
$$

where, to achieve the fourth line, we change order of integration.
On the other hand,

$$
\begin{aligned}
O_2 = & \int_0^1 \left(\int_v^1 \int_v^1 r(v)\, dF(v_2)\, dF(v_1) \right) dG(r(v)) \\
&+ \int_0^1 \left(\int_v^1 \int_0^v \max\{r(v) - v + v_2, 0\} dF(v_2)\, dF(v_1) \right) dG(r(v)) \\
&+ \int_0^1 \left(\int_0^v \int_v^1 \max\{r(v) - v + v_1, 0\} dF(v_2)\, dF(v_1) \right) dG(r(v)) \\
&+ \int_0^1 \left(\int_0^v \int_{v-v_1}^v \max\{r(v) - 2v + v_1 + v_2, 0\} dF(v_2)\, dF(v_1) \right) dG(r(v)) \\
&+ \int_1^2 \left(\int_{v-1}^1 \int_{v-v_1}^1 \max\{r(v) - 2v + v_1 + v_2, 0\} dF(v_2)\, dF(v_1) \right) dG(r(v)).
\end{aligned}
$$

Let us denote the first, second, third, fourth and fifth summands by $O_{2,1}$, $O_{2,2}$, $O_{2,3}$, $O_{2,4}$ and $O_{2,5}$ respectively.
We can write

$$
O_{2,1} = \int_0^1 r(v)\, (1 - F(v))^2\, dG(r(v))
$$

and

$$
\begin{aligned}
O_{2,2} &= \int_0^1 \left(\int_v^1 \int_{v-r(v)}^v (r(v) - v + v_2)\, dF(v_2)\, dF(v_1) \right) dG(r(v)) \\
&= \int_0^1 \left(\int_v^1 ((r(v) - v)(F(v) - F(v - r(v))) + L(v) - L(v - r(v)))\, dF(v_1) \right) dG(r(v)) \\
&= \int_0^1 ((r(v) - v)(F(v) - F(v - r(v))) + L(v) - L(v - r(v)))\, (1 - F(v))\, dG(r(v))
\end{aligned}
$$

and

$$
\begin{aligned}
O_{2,3} &= \int_0^1 \left(\int_0^v \max\{r(v) - v + v_1, 0\}\, (1 - F(v))\, dF(v_1) \right) dG(r(v)) \\
&= \int_0^1 \left(\int_{v-r(v)}^v (r(v) - v + v_1)\, dF(v_1) \right) (1 - F(v))\, dG(r(v)) \\
&= \int_0^1 ((r(v) - v)(F(v) - F(v - r(v))) + L(v) - L(v - r(v)))\, (1 - F(v))\, dG(r(v)) \\
&= O_{2,2}
\end{aligned}
$$

and

$$\begin{aligned}O_{2,4} &= \int_0^1 \left(\int_v^{2v} \max\{r(v) - 2v + t_v, 0\} dH_v(t_v) \right) dG(r(v)) \\&= \int_0^1 \left(\int_{2v-r(v)}^{2v} (r(v) - 2v + t_v)\, dH_v(t_v) \right) dG(r(v)) \\&= \int_0^1 \left((r(v) - 2v)(H_v(2v) - H_v(2v - r(v))) + K_v(2v) - K_v(2v - r(v)) \right) dG(r(v))\end{aligned}$$

and lastly,

$$\begin{aligned}O_{2,5} &= \int_1^2 \left(\int_v^2 \max\{r(v) - 2v + t, 0\} dH(t) \right) dG(r(v)) \\&= \int_1^2 \left(\int_{2v-r(v)}^2 (r(v) - 2v + t)\, dH(t) \right) dG(r(v)) \\&= \int_1^2 \left((r(v) - 2v)(1 - H(2v - r(v))) + K(2) - K(2v - r(v)) \right) dG(r(v)).\end{aligned}$$

Adding all the terms up, we obtain

$$\begin{aligned}O = &\int_0^1 \left(K(v)\, g(r(v))\, r'(v) - (1 - G(r(v)))\, r(v)\, h(v) \right) dv \\&+ \int_0^1 r(v)(1 - F(v))^2 g(r(v))\, r'(v)\, dv \\&+ \int_0^1 2\left((r(v) - v)(F(v) - F(v - r(v))) + L(v) - L(v - r(v)) \right)(1 - F(v))\, g(r(v))\, r'(v)\, dv \\&+ \int_0^1 \left((r(v) - 2v)(H_v(2v) - H_v(2v - r(v))) + K_v(2v) - K_v(2v - r(v)) \right) g(r(v))\, r'(v)\, dv \\&+ \int_1^2 \left(K(v)\, g(r(v))\, r'(v) - (1 - G(r(v)))\, r(v)\, h(v) \right) dv \\&+ \int_1^2 \left((r(v) - 2v)(1 - H(2v - r(v))) + K(2) - K(2v - r(v)) \right) g(r(v))\, r'(v)\, dv.\end{aligned} \tag{7}$$

This problem is now of the form (1)