Core Deviation Minimizing Auctions

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April 23, 2014

Abstract

In a stylized environment with complementary products, we study a class of dominant strategy implementable direct mechanisms and focus on the objective of minimizing the expected surplus from core deviations. For this class of mechanisms, we formulate the core deviation minimization problem as a calculus of variations problem and numerically solve it for some interesting special cases. We then compare the core deviation surplus in the optimal auction (CDMA) to that in Vickrey-Clark-Groves mechanism (VCG) and core-selecting auctions (CSAs). We find that the expected surplus from core deviations can be significantly smaller in CDMA than that in both VCG and CSAs.

JEL Classifications Codes: D44, C71

Keywords: Core, Auctions, Mechanism Design

1 Introduction

When the objects on sale are complements, dominant strategy incentive compatible and ex-post efficient allocation mechanism, namely VCG, produces outcomes that are not always in the core and could therefore yield very low revenues. (Day and Milgrom, 2008 and Ausubel and Baranov, 2010).[1] If the outcome is not in the core then after the auction is over a subgroup of buyers (whose aggregate valuation for some items is greater than the total price paid for them in the auction) would be willing to renegotiate with the seller, or vice versa. This would make the auction outcome unstable. Furthermore, on the same basis, losing buyers could claim that they were ready to pay more but the auction/seller has treated them "unfairly". Although VCG has very strong efficiency and incentive compatibility features, the instability with

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[1] Also, VCG revenues are non-monotonic with respect to the number of bidders or their valuations.

respect to renegotiation and "perceived unfairness," together with the low revenue problem, makes it a weak candidate as a practical allocation mechanism especially when the items are complementary.

Motivated by this observation, Day and Raghavan (2007) and Day and Milgrom (2008) introduced core-selecting auctions (CSAs) as alternatives to VCG.[2] Since then different forms of CSAs have been proposed, but they all share the main features that the allocation is efficient and the payoff vector associated with the auction outcome is always in the core *given that buyers' reports are their true valuations*.[3] The core is defined as the set of feasible allocations that cannot be improved upon by any coalition of buyers (who use their initial endowments). Since the buyers lack the property rights on the objects, a deviating coalition necessarily includes the seller.[4]

In *complete information* settings, CSAs have been shown to have superior properties in terms of revenue while ensuring the noncollusive behavior. Yet, the results obtained in this literature are not readily applicable to environments with private information. This follows because buyers' reported valuations could not be guaranteed to be truthful as CSAs ignore the incentive constraints.

Recently, in a simple stylized environment with *private information*, Goeree and Lien (2012) and Ausubel and Baranov (2010) compare VCG and CSAs with respect to revenue, efficiency, and the distance from the core. Their results demonstrate that VCG and CSAs cannot be generally ranked in terms of above measures, and their relative performances depend on the specifications of buyers' value distributions.[5]

More specifically, Goeree and Lien (2012) establish the following strong impossibility result: when VCG outcome is not in the core, no Bayesian incentive-compatible core-selecting auction exists. That is, for any Bayesian incentive compatible auction,

[2]There is a small but growing literature on CSAs. See, also, Baranov (2010), Cramton (2013), Day and Cramton (2012), Erdil and Klemperer (2010), Lamy (2010) and Sano (2010).

[3]Ausubel and Baranov (2010) review and illustrate the payment rules in Nearest-Vickrey, Proxy, Proportional pricing, Nearest-Bid and First-price package auctions.

[4]The core property does not imply collusion-proofness (in the sense of Marshall and Max (2007) and Ausubel and Milgrom (2002)). For instance, the outcome in a single object second-price auction is in the core, but bidders can collude by bidding less in the auction and reallocating among themselves, or the seller and the second-highest bidder can both benefit if the seller pays the bidder a few pennies to bid a dollar higher. Moreover, in the stylized model where two objects are allocated among two local and one global bidders via VCG, two local bidders can bid artificially high, win their respective items, and pay zero. These kinds of deviations are not considered in our formulation. These observations imply that our results should be interpreted with caution, as we only allow for deviations in which the seller and subset of buyers could divide the available social surplus among themselves and we do not consider their joint manipulation within the given mechanism. In this sense, we consider only the mechanisms that are strategy-proof, but not necessarily those that are group strategy-proof.

[5]See Back and Ott (2013) for a discussion of equilibria in minimum revenue CSA.

the distance of the outcome from the core as measured by the *expected surplus from core deviations* is strictly positive.

In the same stylized environment, we look for the second best and ask the following mechanism design question: Among dominant strategy incentive compatible direct mechanisms, which mechanism achieves an outcome that is closest to the core? By minimizing the expected surplus from core deviations, the optimal mechanism (namely, CDMA) minimizes the incentives for post-auction renegotiations and hence mitigates the perceived unfairness and instability problems to the best extent possible given that buyers must be induced to (*individually*) report truthfully.

For a class of dominant strategy incentive compatible direct mechanisms, we represent this mechanism design problem as a standard calculus of variations problem and prove that the optimal mechanism should "favor the global buyer" in the sense that if global buyer's value is greater than the aggregate values of the local buyers, then the global buyer is always awarded both items. While an analytical closed-form solution to the calculus of variations problem turns out to be difficult to obtain, we numerically solve it for two interesting cases. First, if buyers' value distributions are all uniform, then, interestingly, optimal numerical solution is virtually equivalent to VCG. Yet in another case, the optimal mechanism performs significantly better than both VCG and core selecting auctions.

2 Model

Two goods, goods 1 and 2, are to be allocated among two local buyers, buyers 1 and 2, and one global buyer, buyer 3. Local buyer i has a positive valuation only for good i and global buyer has a positive valuation only for the bundle. Local buyer i's value v_i is distributed over $[0,1]$ according to a distribution function F and global buyer's value for the bundle v_3 is distributed over $[0,2]$ according to a distribution function G. We assume that distributions F and G are atomless, continuous and differentiable. The profile of values are denoted by $\mathbf{v} \equiv (v_1, v_2, v_3)$.

Using revelation principle, we focus on truthful direct mechanisms of the form $\langle Q, T \rangle$ where Q denotes the allocation rule and T denotes the payment rule. We assume that the mechanism is deterministic and that a buyer's probability of getting his respective item (or bundle) increases with his own value. Note that the seller would not want to allocate only one object to the global buyer or both objects to one of the local buyers. Therefore, we can write $Q = (Q_l, Q_g)$ where Q_l and Q_g are probabilities of winning for local buyers and global buyer, respectively. Of course, $Q_l + Q_g = 1$. Given the monotonicity of the allocation rule, we can define the allocation rule using $\widehat{r}(v_1, v_2)$ such that

$$Q_l(\mathbf{v}) = 1 \text{ if and only if } v_3 < \widehat{r}(v_1, v_2) \tag{1}$$

where $\widehat{r}$ is an increasing function of its arguments. To simplify further, we suppose that

$$\widehat{r}(v_1, v_2) = r(v_1 + v_2) \tag{2}$$

where r satisfies[6]

$$r' > 0, \quad r(0) = 0, \quad r(2) = 2. \tag{3}$$

3 Surplus from core deviations

Given the environment and the class of dominant strategy incentive compatible direct mechanisms described in the previous section, we consider the mechanism that minimizes the expected surplus from core deviations. Since a deviating group includes the seller, it is composed either of the seller and the two local buyers or of the seller and the global buyer. Then, the expected surplus from core deviations can be defined as

$$\int_0^1 \int_0^1 \int_0^{r(v_1+v_2)} \max\{v_3 - T_1(\mathbf{v}) - T_2(\mathbf{v}), 0\} dG(v_3)\, dF(v_2)\, dF(v_1)$$
$$+ \int_0^1 \int_0^1 \int_{r(v_1+v_2)}^2 \max\{v_1 + v_2 - T_3(\mathbf{v}), 0\} dG(v_3)\, dF(v_2)\, dF(v_1). \tag{4}$$

Our first result establishes that for dominant strategy implementable mechanisms that satisfy (1)-(3), the payment rule takes a simple and intuitive form.

Proposition 1 *Consider a dominant strategy implementable mechanism which satisfies (1)-(3). Then the payment rule for local buyer i, where $\{i, j\} = \{1, 2\}$ and $j \neq i$, is given by*

$$T_i(\mathbf{v}) = \max\left\{r^{-1}(v_3) - v_j, 0\right\} \tag{5}$$

if $v_i > \max\{r^{-1}(v_3) - v_j, 0\}$ and zero otherwise and that for global buyer is given by

$$T_3(\mathbf{v}) = r(v_1 + v_2) \tag{6}$$

if $v_3 > r(v_1 + v_2)$ and zero otherwise.

[6] The assumption in (2) is essential for the results we present in this paper. While being somewhat restrictive, it is at the same time quite reasonable as the allocation rule treats local buyers symmetrically and takes as inputs the value of the global buyer and the aggregate values of local buyers. When information is private, this assumption is satisfied in equilibrium by VCG but not by CSAs. In equilibrium, CSAs' allocation rules also treat local buyers symmetrically, but they positively favor the global buyer in a more complex way than CDMA does. This follows because CSA allocations depend on the individual values of the local buyers and not simply on their aggregate.

Proof. Dominant strategy incentive compatibility requires that

$$Q_i(v_i, v_{-i}) v_i - T_i(v_i, v_{-i}) \geq Q_i(v_i', v_{-i}) v_i - T_i(v_i', v_{-i}) \quad (7)$$

for all $i \in \{1, 2, 3\}$, v_i, v_i' and v_{-i}. Then, using Myerson's technique, we can write the payments as

$$T_i(v_i, v_{-i}) = Q_i(v_i, v_{-i}) v_i + T_i(0, v_{-i}) - \int_0^{v_i} Q_i(t, v_{-i})\, dt. \quad (8)$$

where

$$Q_i(v_i, v_{-i}) = \begin{cases} 1 & \text{if } v_i > \max\{r^{-1}(v_3) - v_j, 0\} \\ 0 & \text{if } v_i < \max\{r^{-1}(v_3) - v_j, 0\} \end{cases} \quad (9)$$

for $i, j = 1, 2$ and $j \neq i$ and

$$Q_3(v_3, v_1, v_2) = \begin{cases} 1 & \text{if } v_3 > r(v_1 + v_2) \\ 0 & \text{if } v_3 < r(v_1 + v_2). \end{cases} \quad (10)$$

One can argue that at the optimum, $T_3(0, v_1, v_2) = T_1(0, v_{-1}) = T_2(0, v_{-2}) = 0$.[7] Then, substituting (9) and (10) into (8) yields (5) and (6). Hence, the result follows.

■

Notice in Proposition 1 that each player's payment is independent of his own valuation. More importantly, when $r(v) = v$ the payment rules (5) and (6) yield VCG payments. Thus, one could interpret these payments to be "distortion by r" of the VCG payments. Note also that r not only distorts the payments but also distorts the allocations away from VCG.

4 Calculus of variations problem

In this section we establish that the optimal mechanism must favor the global buyer and that the mechanism design problem can be transformed into a standard calculus of variations problem.

Proposition 2 *Core deviation minimizing auction must satisfy*

$$r(v) \leq v. \quad (11)$$

[7] Note that $T_i(0, v_{-i})$ cannot be positive because otherwise individual rationality constraint of buyer i of type 0 is violated. If, on the other hand, $T_i(0, v_{-i})$ is strictly negative, then we can subtract that number from the payments of buyer i, which will not change i's incentive constraints, but decrease the value of the objective function.

Proof. First, we substitute (5) and (6) into (4) and change the order of integrations to obtain

$$\int_0^2 \left(\begin{array}{c} \int_{[r^{-1}(v_3)-1]_+}^1 \int_{[r^{-1}(v_3)-v_1]_+}^1 \left[\begin{array}{c} v_3 \\ -\left[r^{-1}(v_3)-v_1\right]_+ \\ -\left[r^{-1}(v_3)-v_2\right]_+ \end{array} \right]_+ dF(v_2)\, dF(v_1) \\ +\int_0^{[r^{-1}(v_3)]^-} \int_0^{[r^{-1}(v_3)-v_1]^-} \left[v_1+v_2-r(v_1+v_2)\right]_+ dF(v_2)\, dF(v_1) \end{array} \right) dG(v_3).$$

where $[x]_+ = \max\{x, 0\}$ and $[x]^- = \min\{x, 1\}$.

After changing the variable $r^{-1}(v_3)$ to v, we can write the objective function as

$$\min_r \int_0^2 \left(\begin{array}{c} \int_{[v-1]_+}^1 \int_{[v-v_1]_+}^1 \left[r(v) - [v-v_1]_+ - [v-v_2]_+\right]_+ dF(v_2)\, dF(v_1) \\ +\int_0^{[v]^-} \int_0^{[v-v_1]^-} \left[v_1+v_2-r(v_1+v_2)\right]_+ dF(v_2)\, dF(v_1) \end{array} \right) dG(r(v)). \tag{12}$$

Next, we show that the optimal mechanism "favors the global buyer."

Suppose $r(v) > v$ on some interval (c, d), then consider the function $\widetilde{r}$ which satisfies $\widetilde{r}(v) = v$ for $v \in (c, d)$, and $\widetilde{r}(v) = r(v)$ for $v \notin (c, d)$. We argue that $\widetilde{r}$ will achieve a smaller value than r at the objective function. If $v_1 + v_2 \in (c, d)$, then $[v_1 + v_2 - r(v_1 + v_2)]_+$ is zero, and decreasing r in this interval up to $r(v) = v$, does not change the value of this term. Moreover, decreasing r in this interval makes $\left[r(v) - [v-v_1]_+ - [v-v_2]_+\right]_+$ smaller. Hence this decrease has no cost, but only benefit. Hence r has to satisfy $r(v) \le v$ at the optimal solution. ■

Note that local buyers gain from a core deviation if the global buyer is allocated the bundle at a price less than their aggregate values. Similarly, the global buyer gains from a core deviation if both goods are allocated to local buyers at an aggregate price less than his value. Note also that a decrease in $r(v)$ would increase the chance that the global buyer wins and decrease the price he pays when he does so. Hence this change would increase the expected payoff of a deviation by the coalition of 'seller and local buyers' and decrease the expected payoff of a deviation by the coalition of 'seller and global buyer'. The converse also holds as we increase $r(v)$. So, we can trade off the payoff to deviation on one side or the other by increasing or decreasing $r(v)$. Since these payoffs are bounded below at 0, if one side is already making no gains from deviation, it will be optimal to (weakly) shift the bias away from it.

Now consider the VCG case, i.e. $r(v) = v$. If global buyer wins then it pays the sum of local buyers' bids. So local buyers have nothing to gain from deviation (assuming they are honest, which they should be). If local buyers win, each pays the global buyer's bid minus the bid of the other local buyer. The sum of the prices is $T_1(v) + T_2(v) = v_3 - (v_1 + v_2 - v_3)$. Since $v_1 + v_2 > v_3$, when local buyers win in the VCG case, it is *always* the case that a losing global buyer would gain from deviation. So, compared with VCG, it is optimal to (weakly) shift the bias away from local buyers towards global buyer, i.e. reduce $r(v)$ relative to $r(v) = v$. Thus, Proposition 2 holds.[8]

Using Proposition 2, the core deviation minimization problem can be summarized as (12) subject to (3) and (11). Note that this problem is already a calculus of variations problem as the objective is to optimize an integral with respect to a function, (here r), where the integrand involves this function and its derivative (namely, via $dG(r(v))$). The following proposition simplifies the objective function by eliminating the integrals in the integrands of (12) to transform it into a standard calculus of variations problem.

Proposition 3 *The core deviation minimization problem can be written as a standard calculus of variations problem of the form*

$$\min_r \int_0^2 A(r(v), r'(v), v)\, dv. \tag{13}$$

Proof. The proof of this proposition is in the online Appendix (available at `http://www.hadiyektas.net/HY-CDMA_Online_Appendix.pdf`) ■

5 Numerical results

For two pairs of distributions, we solve problem (13) using the Newton-Cotes formulas: We divide the interval of integration to n pieces, and convert $r(v)$ function to a vector of $n+1$ elements (this corresponds to $n+1$ decision variables). We use the following approximation formula

$$\int_a^b \psi(x)\, dx \approx \sum\nolimits_{i=0}^{n} \psi\left(a + i\frac{b-a}{n}\right)\frac{b-a}{n}$$

to evaluate the integrals in the objective function. We then use MATLAB's `fmincon` function to optimize this discretized objective.

Suppose first that both F and G are uniform: $F(v) = v$, $G(v) = \frac{v}{2}$. Then, interestingly, the numeric solution for this case (which is the case that Goeree and Lien

[8]The intuition from Proposition 2 and equations (5) – (6) suggest that CDMA could be implemented or, at least, approximated using a "Vickrey preference (or handicap) auction" - a modified Vickrey auction - that *nonlinearly* deflates the bids of the local bidders to determine the winner and the price winning global bidder pays and *nonlinearly* inflates the bid of the global buyer (while keeping the bids of locals the same) to determine the price winning local buyers pay.

(2012) analyzed) is virtually the Vickrey auction with $r(v) = v$ and expected core deviation surplus equal to 0.125. Suppose next that G is uniform but $F(v) = v^2$. Then, optimal $r(v)$ is nonlinear with expected core deviation surplus equal to 0.1754. For this case, core deviation surplus for Vickrey auction, $r(v) = v$, is 0.2222. Hence, the core deviation minimizing auction (CDMA) as compared to Vickrey auction improves the objective approximately by 21%.

This result is in line with the discussion above following Proposition 2. In VCG, there is no gain from deviation for local buyers when the global buyer wins both items. Therefore, we do not expect CDMA being different from VCG when the likelihood of this event is relatively high. When the likelihood of this event becomes sufficiently low, the optimal CDMA deviates from VCG in the direction of favoring the global buyer (by discounting the bids of local buyers and inflating their payments). Favoring local buyers is counterproductive since their win is the only reason for VCG problems.

Using equilibrium characterizations in Ausubel and Baranov (2010), we also compute the expected surplus from core deviations for different forms of core selecting auctions:

		$CDMA$	VCG	$CSAs$		
				$Proxy$	$Nearest\ VCG/$ $Proportional$	$NearestBid$
$F(v) = v$	$G(v) = \frac{v}{2}$	0.1250	0.1250	0.1961	0.2010	0.2500
$F(v) = v^2$	$G(v) = \frac{v}{2}$	0.1754	0.2222	0.2196	0.2414	0.3487

Table1: Core deviation surplus for various mechanisms

Note that core deviation surplus in CDMA is substantially smaller than those for core selecting auctions.

We have run our algorithm for other specifications of F and G, where both assume the form of power function with various values for the exponent. It turns out that the results are not qualitatively different. When F becomes more convex (or stronger in a first-order stochastic sense) for a fixed G, the optimal r gets away from $r(v) = v$, and when G becomes more concave (or weaker in a first-order stochastic sense) for a fixed F, similarly the optimal r gets away from $r(v) = v$. This can be observed in Figures 1 and 2:

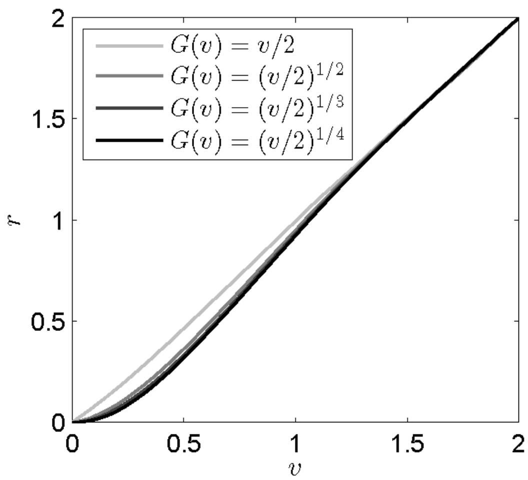

Figure 1: Optimal $r(v)$ for $F(v) = v$ and $G(v) = \left(\frac{v}{2}\right)^{\frac{1}{k}}$ for $k = 1, 2, 3, 4$.

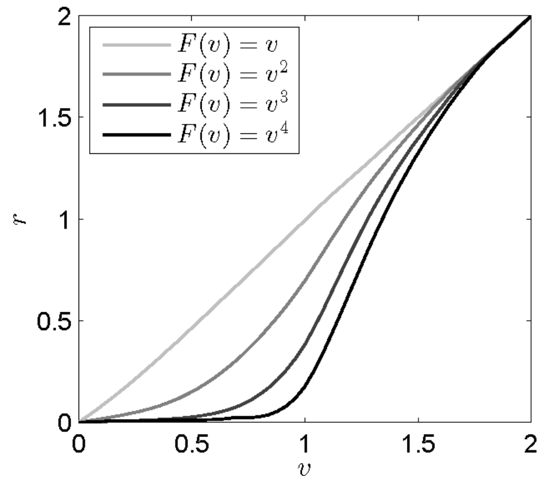

Figure 2: Optimal $r(v)$ for $G(v) = \frac{v}{2}$ and $F(v) = v^k$ for $k = 1, 2, 3, 4$.

These numerical results led us to conjecture the following. Optimal allocation rule that minimizes the distance from core outcomes favor global buyers more and more when local (global) buyers' valuations become stronger (weaker) in the first-order stochastic sense. More formally,

Conjecture 1 *Let $r_{F,G}(v)$ denote the core-deviation minimizing allocation rule for distributions F and G. If $\widetilde{F}(v) \leq F(v)$ and $\widetilde{G}(v) \geq G(v)$ for all possible v, then $r_{\widetilde{F},\widetilde{G}}(v) \geq r_{F,G}(v)$.*

6 Conclusion and Discussion

In a stylized model with complementarities, we ask how close we can get to the core among the strategy proof mechanisms. If the allocation function of the mechanism compares the sum of local buyers' values and global buyer's value, it turns out that this problem can be reduced to a calculus of variations problem which can be solved, at least numerically. We consider two specific examples. When both F and G are uniform, it turns out that the core deviation minimizing auction is virtually the Vickrey auction, whereas if we consider $F(v) = v^2$ and $G(v) = \frac{v}{2}$, the optimal solution improves the objective by approximately 21% as compared to Vickrey auction, and about as much or more when compared to CSAs. We have also shown that in general the global buyer should be favored by the optimal mechanism at the expense of the local buyers.

It is worthwhile to note that both CSA and CDMA attempt to fix the noncore property of VCG using two different approaches: CSAs eliminate visible "perceived unfairness" with respect to reported values while CDMA minimizes distance from "the true core under incentive-compatibility." Interestingly, both approaches result in the same recipe of disfavoring complementary (local) buyers in order to get closer to the core outcomes! Secondly, our paper brings a new theoretical argument in favor of CSAs by pointing out that in some environments CSAs such as Proxy rule can end up being closer to CDMA than VCG.

Although we worked on the stylized model, core deviation minimizing auction problem can be defined for more general settings. The same principles apply with more items, buyers, and combinatorial valuations. More specifically, we can consider an auctioneer who has k different items to sell and n buyers ($\{1, ..., n\} = N$) who have combinatorial valuations over these items. For this general case, the core deviation minimizing auction minimizes the expected value corresponding core deviation surpluses (which is defined as the maximum welfare gain a coalition involving seller can get as compared to their current social welfare) subject to the standard dominant strategy incentive compatibility and individual rationality constraints.

Acknowledgements: We would like to thank the associate editor and two anonymous referees for their insightful and constructive comments that immensely improved our paper. We are grateful to Bilkent University for hosting us during the early stages of this research. We also thank Gokhan Apaydin and Mohamad Maleki for discussions about numerical optimization and Ersin Korpeoglu for excellent research assistantship. Hafalir acknowledges financial support from National Science Foundation grant SES-1326584.

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